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The Packing Problem for Projective Geometries over $GF(3)$ with Dimension Greater Than Five

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An (n, d) set in the projective geometry $PG(r, q)$ is a set of n points, no d of which are dependent. The packing problem is that of finding $n(r, q, d)$, the largest size of an (n, d) set in $PG(r, q)$. The packing problem for $PG(r, 3)$ is considered. All of the values of $n(r, 3, d)$ for $r \leqslant 5$ are known. New results for $r = 6$ are $n(6, 3, 5) = 14$ and $20 \leqslant n(6, 3, 4) \leqslant 31$. In general, upper bounds on $n(r, q, d)$ are determined using a slightly improved sphere-packing bound, the linear programming approach of coding theory, and an orthogonal (n, d) set with the known extremal values of $n(r, q, d)$—values when r and d are close to each other. The BCH constructions and computer searches are used to give lower bounds. The current situation for the packing problem for $PG(r, 3)$ with $r \leqslant 15$ is summarized in a final table.

1. THE PACKING PROBLEM AND $n(r, q, d)$

For integers k and q, where q is a power of a prime, $GF(q)$ denotes the finite field with q elements, and $GF(q)^k = \{\mathbf{c} = (c_1, c_2, ..., c_k)\colon c_i \in GF(q), i = 1, 2, ..., k\}$ the vector space of k tuples with entries in $GF(q)$. Define two sets $\mathscr{P}$ and $\mathscr{L}$ as

$$\mathscr{P} = \{p \subseteq GF(q)^{r+1}: p \text{ is a one-dimensional subspace}\},$$

$$\mathscr{L} = \{l \subseteq GF(q)^{r+1}: l \text{ is a two-dimensional subspace}\}.$$

Then with the usual incidence $(\mathscr{P}, \mathscr{L})$ forms a projective geometry of order q with projective dimension r, and is denoted by $PG(r, q)$.

Each projective point p, as a subspace of dimension 1, is spanned by a single nonzero vector. So, if $\mathbf{a} = (a_1, a_2, ..., a_{r+1}) \in p$, then $p = \{\alpha\mathbf{a}\colon \alpha \in GF(q)\}$. In practice, the projective point p, as a subspace, is identified with some nonzero vector it contains, and in this case the point is denoted by $\mathbf{p}$.

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Let n and d be integers with $2 \leqslant d \leqslant r+1$. A set $C \subseteq PG(r, q)$ is an (n, d) *set* provided $|C| = n$, and no d points of C are contained in a $(d-2)$ flat. Alternatively, no d of the vectors of C are linearly dependent. Let $n(r, q, d)$ be the largest n such that an (n, d) set C exists in $PG(r, q)$. The *packing problem* for $PG(r, q)$ concerns determining the values of $n(r, q, d)$ for $2 \leqslant d \leqslant r+1$, and an *optimal* (n, d) set which has $n = n(r, q, d)$.

Other terminology for the packing problem that has appeared include:

(1) n *arc* $\subseteq PG(r, q)$ or $n_r$arc is an $(n, r+1)$ set $\subseteq PG(r, q)$ (Segre [23]),

(2) n *cap* $\subseteq PG(r, q)$ is an $(n, 3)$ set (Hill [15]),

(3) n *set of kind* s is an $(n, s+1)$ set with the property that there is a set of $s+2$ dependent points (Barlotti [1]),

(4) *d-independent set of size* $n \subseteq PG(r, q)$ is an (n, d) set (Dowling [8]).

Also, the value of $n(r, q, d)$ is often denoted by $m_d(r+1, q)$ or $m_{r,q}^{d-1}$.

A number of properties of $n(r, q, d)$ have been noticed by many authors:

(1) $n(r, q, 2) = |PG(r, q)| = (q^{r+1} - 1)/(q-1)$,

(2) $n(r, q, d) \geqslant n(r-1, q, d) + 1$,

(3) $n(r, q, d) \geqslant n(r, q, d+1)$,

(4) $n(r, q, d) \leqslant n(r-1, q, d-1) + 1$,

(5) $n(r, q, d) = n(r-1, q, d-1) + 1$, for $q = 2$ and d odd.

In this article, we consider the packing problem for $PG(r, 3)$. All of the values of $n(r, 3, d)$ for $r \leqslant 5$ are given in [17], so we considered $r \geqslant 6$. We first review the connection of the packing problem to linear error-correcting codes given in [4, 16].

Let $C = \{x_1, x_2, \ldots, x_n\} \subseteq PG(r, q)$ be an (n, d) set. Two linear codes will be associated with C. The *matrix of* C is an n by $r+1$ array $H(C)$ which has rows $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$,

$$H(C) = \begin{bmatrix} \mathbf{x}_1 \\ --- \\ \mathbf{x}_2 \\ --- \\ \vdots \\ --- \\ \mathbf{x}_n \end{bmatrix}.$$

If C is not contained in any hyperplane of $PG(r, q)$, then $H(C)$ has rank $r+1$. Thus, the columns of $H(C)$, as code words of length n, generate an $(n, r+1)$ linear code $\mathscr{C}$. Second, $H(C)$ is the parity check matrix for an $(n, n-r-1)$ linear code $\mathscr{C}^{\perp}$.

That C forms an (n, d) set implies $\mathscr{C}^{\perp}$ has minimum weight greater than d. A code word $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathscr{C}^{\perp}$ satisfies $\mathbf{c}H(C) = \mathbf{0}$; that is, $\sum_{i=1}^{n} c_i \mathbf{x}_i = \mathbf{0}$. So, $\{\mathbf{x}_i \colon c_i \neq 0\}$ is a dependent set of vectors of size $\mathrm{wt}(\mathbf{c})$. Since no d vectors of $\mathbf{C} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are dependent, $\mathrm{wt}(\mathbf{c}) \geqslant d+1$. Conversely, the rows of the parity check matrix of an $(n, n-r-1)$ linear code with minimum weight greater than d, for $d \geqslant 2$, form an (n, d) set contained in $PG(r, q)$.

Finally, if the minimum weight of $\mathscr{C}$ is $d^{\perp}+1$, for $d^{\perp} \geqslant 2$, then the columns of an $n-r-1$ by n generator matrix of $\mathscr{C}^{\perp}$ are distinct points of $PG(n-r-2, q)$ and form an $(n, d^{\perp})$ set denoted by $C^{\perp}$. In this case, $C^{\perp} \subseteq PG(n-r-2, q)$ is called an *orthogonal* $(n, d^{\perp})$ set of C.

2. The Coding Bounds

These upper bounds correspond to upper bounds of coding theory for the $(n, n-r-1)$ linear code $\mathscr{C}^{\perp}$ of minimum weight $d+1$ which can correct $e = [d/2]$ errors. They are Johnson-type bounds [18]. See [9] for details. Here $[x]$ denotes the greatest integer $\leqslant x$.

d Odd

If $C \subseteq PG(r, q)$ is an (n, d) set, then n must satisfy

$$q^{r+1} \geqslant \sum_{i=0}^{e} \binom{n}{i} (q-1)^i + \frac{\binom{n}{e+1} (q-1)^{e+1}}{\left[\dfrac{n}{e+1}\right]}. \tag{B1}$$

If $[n/(e+1)] = n/(e+1)$, then (B1) equals a result of Rao [22],

$$q^{r+1} \geqslant \sum_{i=0}^{e} \binom{n}{i} (q-1)^i + \binom{n-1}{e} (q-1)^{e+1}, \tag{B2}$$

but the bound on n implied by (B1) improves that implies by (B2) in some cases.

d Even

If $C \subseteq PG(r, q)$ is an (n, d) set, then n must satisfy

$$q^{r+1} \geqslant \sum_{i=0}^{e} \binom{n}{i} (q-1)^i + \frac{\binom{n}{e+1}(q-1)^{e+1} - \binom{n}{e}(q-1)^e \left[\dfrac{(n-e)(q-1)}{e+1}\right]}{\left[\dfrac{n(q-1)}{e+1}\right]}. \qquad \text{(B3)}$$

If $[(n-e)(q-1)/(e+1)] = (n-e)(q-1)/(e+1)$, then (B3) reduces to the sphere packing bound

$$q^{r+1} \geqslant \sum_{i=0}^{e} \binom{n}{i} (q-1)^i. \qquad \text{(B4)}$$

The bound implied by (B3) improves that implied by (B4) in some cases.

3. The Linear Program

In this section, the linear programming approach of coding theory is used to derive upper bounds on $n(r, q, d)$. Because the codes involved are linear, constraints that improve the optimal value of the linear program can be added to the program. Examples for the case $q = 3$ are considered, and two new results are $n(8, 3, 6) \leqslant 23$ and $n(10, 3, 8) \leqslant 20$.

Let $C \subseteq PG(r, q)$ be an (n, d) set, $\mathscr{C}$ the associated $(n, r+1)$ linear code, and $\mathscr{C}^\perp$ the associated $(n, n-r-1)$ linear code with minimum distance $d+1$. Suppose $\mathscr{C}$ and $\mathscr{C}^\perp$ have respective weight enumerators $(A_0 = 1, A_1, ..., A_n)$ and $(B_0 = 1, \quad B_1 = 0, ..., B_d = 0, \quad B_{d+1}, ..., B_n)$, then the MacWilliams identities [19, Chap. 5]) are

$$A_i = 1/(q^{n-r-1}) \sum_{j=0}^{n} B_j K_i(j), \qquad i = 0, 1, 2, ..., n,$$

where $K_i(x)$ is the Krawtchouk polynomial of degree i, namely,

$$K_i(x) = \sum_{k=0}^{i} (-1)^k (q-1)^{i-k} \binom{x}{k} \binom{n-x}{i-k}.$$

Notice that $K_i(0) = (q-1)^i\binom{n}{i}$. Thus, the weight enumerator of the linear code $\mathscr{C}^\perp$ is a feasible solution of the linear program,

$$\text{maximize } 1 + \sum_{j=d+1}^{n} x_j,$$

subject to

$$\sum_{j=d+1}^{n} x_j K_i(j) \geqslant -(q-1)^i \binom{n}{i}, \qquad i = 1, 2, \ldots, n,$$

$$x_j \geqslant 0, \qquad j = d+1, \ldots, n. \qquad \text{(LP1)}$$

See Delsarte [7], where linear programming is applied to general codes. If $\mathscr{C}$ has minimum weight $d^\perp + 1$, then the linear program becomes

$$\text{maximize } 1 + \sum_{j=d+1}^{n} x_j,$$

subject to

$$\sum_{j=d+1}^{n} x_j K_i(j) = -(q-1)^i \binom{n}{i}, \qquad i = 1, 2, \ldots, d^\perp,$$

$$\geqslant -(q-1)^i \binom{n}{i}, \qquad i = d^\perp + 1, \ldots, n,$$

$$x_j \geqslant 0, \qquad j = d+1, \ldots, n. \qquad \text{(LP2)}$$

An optimal value B^* of (LP1) or (LP2) gives an upper bound on the size of $\mathscr{C}^\perp$. Suppose k^* is chosen so that $q^{k^*} \leqslant B^* \leqslant q^{k^*+1}$, then $|\mathscr{C}^\perp| = q^{n-r-1} \leqslant q^{k^*}$; that is, $n-r-1 \leqslant k^*$ or $r^* \equiv n - k^* - 1 \leqslant r$. Furthermore, $n(r^*-1, q, d) \leqslant n-1$. The best upper bound on $n(r, q, d)$ is obtained by applying the linear program to successive values of n until an n^* is obtained such that $r^*(n^*) < r^*(n^*+1)$. Here $r^*(n)$ denotes the lower bound obtained when the length n is used in (LP1) or (LP2). In this case, $n(r^*, q, d) \leqslant n^*$.

Additional constraints can be derived by using the linearity of the codes involved. When the constraints are added to (LP1) or (LP2), a decrease in the optimal value B^* of the linear program may result. Any decrease in the optimal value yields an improved lower bound on r exactly when the new optimal value B^{**} satisfies $B^{**} < q^{k^*} \leqslant B^* \leqslant q^{k^*+1}$.

In the following, let $A(n, D, w)$ be the maximum number of code words in any binary code of length n, constant weight w, and minimum distance D. If $D(t, k, v)$ is the maximum number of k subsets of a v set V, such that every t

subset of V is contained in at most one k set, then $D(t, k, v) = A(v, 2(k-t+1), k)$. For a table of upper bounds on $A(n, D, w)$ see Best *et al.* [2]. The additional constraints that are derived have two forms, support upper bounds on B_{d+1} and elimination upper bounds on B_{d+1}.

Support Upper Bounds on B_{d+1}

Let $C \subseteq PG(r, q)$ be an (n, d) set and $\mathscr{C}^{\perp}$ the associated $(n-r-1)$ linear code. The idea is that two words of minimum weight $d+1$ have supports that can have only a limited number of positions in common.

LEMMA 3.1. *Let* $\mathbf{b}$ *and* $\mathbf{c}$ *be two* i *tuples with entries in* $GF(q)^* = GF(q)\backslash\{0\}$, *then for some* $\alpha \in GF(q)^*$, $\mathrm{wt}(\mathbf{b} - \alpha\mathbf{c}) \leqslant (q-2)\, i/(q-1)$.

Proof. Let $\mathbf{b} = (b_1, b_2, ..., b_i)$ and $\boldsymbol{c} = (c_1, c_2, ..., c_i)$, and define $S_\alpha(\mathbf{b}, \boldsymbol{c}) = \{j: b_j = \alpha c_j\}$. Then $(S_\alpha(\mathbf{b}, \boldsymbol{c}): \alpha \in GF(q)^*)$ partitions $\{1, 2, ..., i\}$ so that there exists $\alpha \in GF(q)^*$ such that $|S_\alpha(\mathbf{b}, \boldsymbol{c})| \geqslant i/(q-1)$. Thus, $\mathrm{wt}(\mathbf{b} - \alpha\boldsymbol{c}) \leqslant i - i/(q-1) = (q-2)\, i/(q-1)$.

LEMMA 3.2. *Suppose* $\mathbf{b}$ *and* $\boldsymbol{c}$ *are two words of minimum weight* $d+1$ *of* $\mathscr{C}^{\perp}$, *then* $|\mathrm{sup}(\mathbf{b}) \cap \mathrm{sup}(\boldsymbol{c})| \leqslant (q-1)(d+1)/q$.

Proof. Let $i = |\mathrm{sup}(\mathbf{b}) \cap \mathrm{sup}(\boldsymbol{c})|$, then, by Lemma 3.1, there is an $\alpha \in GF(q)^*$ such that the word $\mathbf{b} - \alpha\boldsymbol{c}$ has $\mathrm{wt}(\mathbf{b} - \alpha\boldsymbol{c}) \leqslant 2(d+1-i) + (q-2)\, i/(q-1)$. Because $\mathscr{C}^{\perp}$ is a linear code, $\mathbf{b} - \alpha\boldsymbol{c} \in \mathscr{C}^{\perp}$, and $d+1 \leqslant \mathrm{wt}(\mathbf{b} - \alpha\boldsymbol{c})$. Thus,

$$d+1 \leqslant 2(d+1-i) + (q-2)\, i/(q-1), \qquad qi \leqslant (q-1)(d+1),$$
$$i \leqslant (q-1)(d+1)/q.$$

Lemma 3.1 implies that two words $\mathbf{b}$ and $\boldsymbol{c}$ of $\mathscr{C}^{\perp}$ of minimum weight with the same support correspond to the same *projective code word*; that is, $\mathbf{b} = \alpha\boldsymbol{c}$ for some $\alpha \in \boldsymbol{c}$. Lemma 3.2 implies that the number of projective code words is bounded above by $D(t, d+1, n)$, where $t-1 = [(q-1)(d+1)/q]$. From this, the *support bound* on B_{d+1} is obtained,

$$B_{d+1} \leqslant (q-1)\, A(n, 2(d+2-t), d+1). \qquad \text{(SB)}$$

An approach like the preceding one will yield upper bounds on $B_i, i \geqslant d+1$, as long as two words with weight i and the same support correspond to the same projective code word.

Elimination Upper Bounds on B_{d+1}

In some cases, the support bound can be improved upon. This improved bound on B_{d+1}, called the elimination bound in [9], where it is derived, is

$$B_{d+1} \leqslant \frac{q-1}{\binom{d+1}{e+1}} \left(\binom{n}{e+1} (q-1)^e \left(\left[\frac{n}{e+1} \right] - 1 \right) \right), \quad d \text{ is odd,}$$

(EB)

$$B_{d+1} \leqslant \frac{q-1}{\binom{d+1}{e+1}} \binom{n}{e} (q-1)^{e-1} \left[\frac{(n-e)(q-1)}{e+1} \right], \quad d \text{ is even.}$$

Examples for $q=3$

The linear programming approach is applied to the cases $q=3$, $5 \leqslant d \leqslant 8$, and $15 \leqslant n \leqslant 24$. Table I lists the support bound (SB) and the elimination bound (EB) in each case. The smaller bound is added to the linear program (LP1), and an optimal solution is found using the simplex algorithm on a computer. The optimal values of the objective function, as well as the optimal levels x^*_{d+1} of x_{d+1} are listed in Tables II–V. Included also, are the implied lower bounds on the projective dimension, and the upper bounds on x_{d+1}, so that those additional constraints that are binding can be determined. A binding additional constraint most likely implies a larger optimal value of

TABLE I

Support and Elimination Bounds on B_{d+1} for $q=3$ and $15 \leqslant n \leqslant 24$

	$d=5$		$d=6$		$d=7$		$d=8$	
n	(SB)	(EB)	(SB)	(EB)	(SB)	(EB)	(SB)	(EB)
15	910[a]	728	176	624	176	624	140	693
16	1444	896	312	768	300	1248	312	924
17	1904	1088	488	1088	566	1632	566	1511
18	2856	1632	698	1305	856	2098	850	1942
19	3578	1938	1040	1771	1478	2657	1578	2953
20	5012	2280	1302	2084	2398	4429	2726	3691
21	6384	3192	1656	2736	3416	5472	4728	4560
22	8778	3696	2200	3168	4554	6688	7550	6502
23	10626	4250	3036	4048	6324	8096	11638	7871
24	14168	5667	3572	4626	9108	12144	16864	10794

[a] Here $910 = 2 \cdot A(n, 4, 6)$; values of $A(n, D, w)$ are obtained from Best *et al.* [2].

TABLE II

Linear Programming Results for $q=3$, $d=5$

n	$x_6 \leqslant$	x_6^*	Optimal value	r^*
15	728	623	10735	6
16	896	896	29523	6
17	1088	1088	74945	6
18	1632	1387.2	195055	6
19	1938	1972.2	559047	6
20	2280	2280	1453524	7
21	3192	2814	3913337	7
22	3696	3696	11394719	7
23	4250	4250	29799842	7
24	5667	5227	82893458	7

the objective function if the constraint were to be dropped. The implied upper bounds on $n(r, 3, d)$ include:

(1) $n(6,3,5) \leqslant 19$ (= sphere packing bound; actually $n(6,3,5)=14$),

(2) $n(8,3,6) \leqslant 23$ (sphere packing bound = 25),

(3) $n(10,3,8) \leqslant 20$ (sphere packing bound = 23).

The bound $n(8,3,6) \leqslant 23$ is of particular interest since without the additional constraint, when $n=24$, an optimal value of 15,638,906 is obtained with $x_7{}^* = 4{,}601$. Compare this value to 13,501,273—the optimal

TABLE III

Linear Programming Results for $q=3$, $d=6$

n	$x_7 \leqslant$	x_7^*	Optimal value	r^*
15	176	176	1713	8
16	312	312	4868	8
17	488	488	13493	8
18	698	698	36121	8
19	1040	1040	97224	8
20	1302	1302	253446	8
21	1656	1656	683842	8
22	2200	2200	1807915	8
23	3036	3036	5097637	8
24	3572	3572	13501273	9

TABLE IV

Linear Programming Results for $q = 3$, $d = 7$

n	$x_8 \leqslant$	x_8^*	Optimal value	r^*
15	176	176	542	9
16	300	300	1520	9
17	566	566	4116	9
18	856	856	11208	9
19	1478	1478	33024	9
20	2398	2398	92044	9
21	3416	3416	252043	9
22	4554	4554	668150	9
23	6324	6324	1797083	9
24	9108	9108	5097637	9

value when $x_7 \leqslant 3{,}572$ is added. Since $13{,}501{,}273 < 3^{15} < 15{,}638{,}906$, the additional constraint makes a real difference because without it $r^* = 8$ (instead of $r^* = 9$), and at best $n(8, 3, 6) \leqslant 24$.

4. The Extremal Values of $n(r, q, d)$ and Orthogonal (n, d) Sets

The method of adding points to the base points of $PG(r, q)$ in order to obtain an (n, d) set is described in this section. Many authors have used this approach in order to determine the "extremal" values of $n(r, q, d)$—values when d is "close" to r. The extremal values then can be used to obtain upper

TABLE V

Linear Programming Results for $q = 3$, $d = 8$

n	$x_9 \leqslant$	x_9^*	Optimal value	r^*
15	140	116.31	165	10
16	312	237.71	450	10
17	566	505.14	1352	10
18	850	711.41	3025	10
19	1578	998.11	7073	10
20	2726	1668	20290	10
21	4560	2379	55197	11
22	6502	3086	139201	11
23	7871	4446	372726	11
24	10794	6867	1051876	11

bounds on other values of $n(r, q, d)$ by considering an orthogonal (n, d^-) set. As an example, the bound on $n(8, 3, 6)$ is improved to 20 from 23, which was the linear programming bound.

Gulati [10], Gulati *et al.* [11], Gulati and Kounias [12], and Hamada and Tamari [13, 14] have studied the extremal values of $n(r, q, d)$ by attempting to add points to the $r+1$ base points. Fix r and d and let $B = \{(1, 0, 0,..., 0), (0, 1, 0,..., 0),..., (0, 0, 0,..., 1)\} \subseteq PG(r, q)$. Let $A = \{a_1, a_2,..., a_k\} \subseteq PG(r, q)$ and form $C = B \cup A$. Assume that C is an (n, d) set with $n = r + 1 + k$. If $\mathbf{a}_i = (a_{i1}, a_{i2},..., a_{ir+1})$, then the matrix of C is

$$H(C) = \begin{bmatrix} I_{r+1} \\ ---- \\ H(A) \end{bmatrix},$$

where I_{r+1} is the identity matrix and

$$H(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r+1} \\ a_{21} & a_{22} & \cdots & a_{2r+1} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kr+1} \end{bmatrix}.$$

PROPOSITION 4.1 (Gulati and Kounias [12]). *Let $C = B \cup A$ be an $(r+1+k, d)$ set formed by adding the elements of $A \subseteq PG(r, q)$, $|A| = k$, to the base points $B \subseteq PG(r, q)$, $|B| = r + 1$. Then r, q, d, and k must satisfy*

$$r + 1 \leqslant q(r - d + k) - k + [(q-1)(r-d+k)/(q^{k-1} - 1)]. \qquad \text{(NC)}$$

If k points $A \subseteq PG(r, q)$ can be added to the base points $B = \{b_1, b_2,..., b_{r+1}\} \subseteq PG(r, q)$ such that $C = B \cup A$ is an $(r+1+k, d)$ set, then (NC) must be satisfied. The existence of a $(r+1+k, d)$ set $\subseteq PG(r, q)$ depends on finding a solution of certain inequalities. This portion of the problem is hard and is treated in Gulati *et al.* [11] and Hamada and Tamari [13, 14]. For our purposes we need the upper bounds on $n(r, q, d)$ implied by (NC).

For fixed r and k, the right-hand side of (NC) increases as d decreases. Set $k = 2$ and $d = r + 1$, and decrease d until (NC) is satisfied, say at d_2. Then, for $d_2 + 1 \leqslant d \leqslant r + 1$, $n(r, q, d) = r + 1 + 1$ (equality results because the point $(1, 1,..., 1)$ can be added to the base points). Then, set $k = 3$ and $d = d_2$, and decrease d until (NC) is again satisfied, say at d_3. For $d_3 + 1 \leqslant d \leqslant d_2$, $n(r, q, d) \leqslant r + 1 + 2$ (whether or not equality occurs depends on finding an actual solution). Continue as indicated, but letting $k = 4, 5,...$ to obtain $d_4, d_5,...$ such that for $d_{i+1} + 1 \leqslant d \leqslant d_i$, $n(r, q, d) \leqslant$

$r+1+i$. Eventually k' and $d_{k'}$ are obtained with the property that for $k=k'+1$, (NC) is satisfied at $d=d_{k'}$. For $d \leqslant d_{k'}$, no upper bound on $n(r,q,d)$ is implied by (NC). The values of $n(r,q,d)$ for $d_{k'}+1 \leqslant d \leqslant r+1$ are called the *extremal values* of $n(r,q,d)$, and $\{d\colon d_{k'}+1 \leqslant d \leqslant r+1\}$ is the *extremal region* for d with respect to r.

EXAMPLE 1. Let $q=3$ and $r=100$, then (NC) implies

$$\begin{aligned} n(100,3,d) &= 102 \quad \text{for} \quad 77 \leqslant d \leqslant 101,\\ n(100,3,d) &\leqslant 103 \quad \text{for} \quad 72 \leqslant d \leqslant 76 = d_2,\\ n(100,3,d) &\leqslant 104 \quad \text{for} \quad 70 \leqslant d \leqslant 71 = d_3. \end{aligned}$$

Here, $d_4=69$. If $d=69$ and $k=5$, (NC) becomes $101 \leqslant 103+0$ so that $k'=4$ and $d_{k'}=69$.

If $d \leqslant (q-1)\,r/q$, then a calculation shows that (NC) is satisfied for any $k \geqslant 2$. Since $((q-1)\,k-1)/q > 0$,

$$d \leqslant (q-1)\,r/q \leqslant ((q-1)\,r+(q-1)\,k-1)/q, \qquad r+1 \leqslant q(r-d+k)-k$$

$$r+1 \leqslant q(r-d+k)-k+[(q-1)(r-d+k)/(q^{k-1}-1)]. \quad \text{(NC)}$$

Thus $\{d\colon d_{k'}+1 \leqslant d \leqslant r+1\} \subseteq \{d\colon (q-1)\,r/q < d \leqslant r+1\}$.

The extremal values of $n(r,q,d)$ or their upper bounds are applied to an orthogonal $(n,d^{\perp})$ set in order to obtain upper bounds on other values of $n(r,q,d)$.

PROPOSITION 4.2. *Let $C \subseteq PG(r,q)$ be an (n,d) set and suppose $n(r-1,q,d) \leqslant m$, then an orthogonal $(n,d^{\perp})$ set $C^{\perp} \subseteq PG(n-r-2,q)$ has $d^{\perp} \geqslant n-m-1$.*

Proof. Any code word $\mathbf{c} \in \mathscr{C}$, the $(n,r+1)$ code generated by C, has $\mathrm{wt}(\mathbf{c}) \geqslant n-m$, since, otherwise, there is a hyperplane of $PG(r,q)$ containing more than m points of C, contradicting $n(r-1,q,d) \leqslant m$.

COROLLARY 4.3. *If $n(r,q,d) \geqslant n$ and $n(r-1,q,d) \leqslant m$, then $n(n-r-2,q,n-m-1) \geqslant n$.*

Suppose r,q, and d are fixed and that $n(r-1,q,d) \leqslant m$. In hopes of bounding $n(r,q,d)$ ask the question: Can there be an (n,d) set $C \subseteq PG(r,q)$? Suppose for $r^{\perp}=n-r-2$ and $d^{\perp}=n-m-1$, it is known that $n(r^{\perp},q,d^{\perp}) \leqslant n^{\perp}$. If $n > n^{\perp}$, then there can be no (n,d) set since then Corollary 4.3 implies that $n(r^{\perp},q,d^{\perp}) \geqslant n > n^{\perp}$, a contradiction. The conclusion is then that $n(r,q,d) \leqslant n-1$. Repeat the above in this case, replacing n with $n-1$ so that $r^{\perp}$ and $d^{\perp}$ are replaced by $r^{\perp}-1$ and $d^{\perp}-1$, respectively. Continue until no contradiction is reached.

The success of the method depends on knowing a good upper bound on $n(r^{\perp}, q, d^{\perp})$, and this is the case if $d^{\perp}$ is in the extremal region for d with respect to $r^{\perp}$. The method is illustrated by applying it to the case $q=3$.

EXAMPLE 2. Here, $n(6,3,4) \leqslant 31$ (sphere packing bound (improved) $= 32$) $r=6$, $q=3$, $d=4$, $n=32$, $m=n(5,3,4)=13$ (Hill [17]), $r^{\perp}=24$, $d^{\perp}=18$, but $n(24,3,18)=27$ (Gulati and Kounias [12]). Thus, $n(6,3,4) \leqslant 31$.

TABLE VI

Upper Bounds on $n(r, 3, d)$, $r \leqslant 15$, Implied by the Extremal Values

r \ d	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	4													
3	10	5												
4	20	11	6											
5	56	13	12	7										
6		≤32 ≤31	14	8	8									
7			≤33 ≤32	≤17 ≤11	9	9								
8				≤25 ≤23 ≤20	≤18 ≤12	10	10							
9					≤25 ≤21	≤17 ≤13	11	11						
10						≤23 ≤20 ≤22	12	12	12					
11							14	13	13	13				
12							≤31 ≤18	15	14	14	14			
13							≤41 ≤33	≤29 ≤19	16	15	15	15		
14								≤36 ≤34	≤30 ≤20	16	16	16	16	
15									≤37 ≤35	18	17	17	17	17

Key

SPB	← Sphere Packing Bound
LP	← Linear Programming
EO	← From Extremal Values and Orthogonal Set

EXAMPLE 3. Here, $n(8,3,6) \leqslant 20$ (SPB = 25, LP bound = 23) $r=8$, $q=3$, $d=6$, $m=n(7,3,6) \leqslant 11$ (see Table VI). Then

n	$r^{\perp}$	$d^{\perp}$	$n(r^{\perp}, 3, d^{\perp})$		remark
23	13	11	16	Gulati and	no
22	12	10	15	Kounias,	no
21	11	9	14	[12]	no
20	10	8	$\leqslant 20$	LP bound	okay.

Thus, $n(8,3,6) \leqslant 20$.

Table VI lists the upper bounds on $n(r,3,d)$ for $r \leqslant 15$ that can be derived using an orthogonal $(n, d^{\perp})$ set and the extremal values of $n(r,3,d)$ derived by Gulati *et al.* [11, 12].

5. THE PACKING PROBLEM FOR $PG(r, 3)$

In this section, lower bounds, actual values, and upper bounds for $n(r,3,d)$ with $r \leqslant 15$ are described. Lower bounds are derived from the BCH construction and computer searches. The upper bounds on $n(r,q,d)$ include the sphere packing bounds, the linear programming bounds, and the bounds derived from the extremal values. Finally, Table VII summarizes all the current values of, and bounds for $n(r,3,d)$ with $r \leqslant 15$.

Now we give three BCH codes over $GF(3)$ (Bose and Ray-Chaudhuri [5, 6]), and the corresponding lower bounds on $n(r,3,d)$.

EXAMPLE 1. ($n=13$; $2 \times 13 = 3^3 - 1$; $n(5,3,4) \geqslant 13$; $n(6,3,5) \geqslant 14$). Let β be a primitive 13th root of unity in $GF(3^3)$. Since $x \to 3x$ acts on $\mathbb{Z}_{12}$ decomposing it into orbits

$$(0)(1,3,9)(2,6,5)(4,12,10)(7,8,11),$$

if $m_\alpha(x)$ denotes the minimal polynomial of $\alpha \in GF(3^3)$, then

$$m_\beta(x) = (x-\beta)(x-\beta^3)(x-\beta^9) \quad = x^3 + x^2 + x + 2,$$

say,

$$m_{\beta^2}(x) = (x-\beta^2)(x-\beta^6)(x-\beta^5) \quad = x^3 + x^2 + 2,$$
$$m_{\beta^4}(x) = (x-\beta^4)(x-\beta^{12})(x-\beta^{10}) = x^3 + 2x^2 + 2x + 2,$$
$$m_{\beta^7}(x) = (x-\beta^7)(x-\beta^8)(x-\beta^{11}) \quad = x^3 + 2x + 2.$$

Let $g(x) = m_{\beta^2}(x)\, m_{\beta^7}(x) = x^6 + x^5 + 2x^4 + 2x^2 + x + 1$, then $g(x)$ is the generator polynomial of a $(13, 7)$-linear code $\mathscr{C}^{\perp}$ with designed distance 5

TABLE VII

$n(r, 3, d), r \leqslant 15$

r \ d	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	B 4													
3	B 10	G 5												
4	P 20	GO 11	G 6											
5	H 56	H 13 BCH	X-GO 12	G 7										
6	163 H 112 Z	31 EO 20 CPU	BCH 14	G 8	G 8									
7	488 H 224 Z	56 32 CPU	32 EO	ADD 11	G 9	G 9								
8	1460 H 488 Z	98	57	20 EO 13 BCH	ADD 12	G 10	G 10							
9	4376 H 896 Z	171	99	35	21 EO 14 BCH	ADD 13	G 11	G 11						
10	13124H 1792 Z	297	172	51	36	20 LP	G 12	G 12	G 12					
11	39368H 3584 Z	515	298	73	52	30 24 SD	GK 14	G 13	G 13	G 13				
12	118100 H 7168 Z	892	516	106	74	40	18 EO 17 CPU	GK 15	G 14	G 14	G 14			
13	354296 H	1546	893	153	106	52	33 EO	19 EO 18 CPU	GK 16	G 15	G 15	G 15		
14	1062884 H	2678	1546	221	154	69	53	34 EO 20 BCH	20 EO 19 CPU	G 16	G 16	G 16	G 16	
15	3188648 H	4639	2679	318	221	90	70	45	35 21 BCH	GK 18	G 17	G 17	G 17	G 17

Key

	#	Reference *
Upper bound →	#	*
Actual value →	#	
Lower bound →	#	

Notes

ADD: Add to base points (Gulati [10])
B: Bose [3]
BCH: BCH construction
CPU: Computer result
EO: Extremal values and orthogonal set
G: Gulati, Mck. Johnson, Koehn [11]
GK: Gulati, Kounias [12]
GO: (11,5) Ternary Golay code
H: Hill [15-17]
P: Pellegrino [21]
SD: Self-dual code, Mallows et al. [20]
X-GO: (12,6) Extended Golay code
: Sphere packing bounds (improved)
Z: $n(r+1,3,3) \geq 2n(r,3,3)$

NOTE: Bounds from $n(r+1,3,d) \geq n(r,3,d)+1$ are not listed.

(which is the actual minimum distance). The orthogonal code is the code of a $(13, 4)$ set $C \subseteq PG(5, 3)$; that is, $n(5, 3, 4) \geqslant 13$.

If $\mathscr{C}^{\perp}$ has generating matrix G, let

$$\mathbf{G} = \left[\begin{array}{c|c} 1 & \\ 1 & \\ \vdots & G \\ 1 & \end{array}\right],$$

then, in this case, $\mathbf{G}$ generates a (14, 7)-linear code $\mathscr{C}^{\perp}$, which is called the *extended code*. Although the extended code may not have a larger minimum distance, it can be checked that the minimum distance of $\mathscr{C}^{\perp}$ increases by one to 6. Then the orthogonal code is the code of a (14, 5) set $\mathbf{C} \subseteq PG(6, 3)$; that is, $n(6, 3, 5) \geqslant 14$. The matrix of $\mathbf{C}$ is the matrix

$$H(\mathbf{C}) = \left[\begin{array}{c|cccccc} 1 & & & & & & \\ 1 & & & & & & \\ \vdots & & & H^{\tau} & & & \\ 1 & & & & & & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where H is the matrix of the polynomial $h(x) = (x-1)\, m_{\beta}(x)\, m_{\beta^4}(x) = x^7 + 2x^5 + x^3 + 2x^2 + x + 2$, namely,

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hill [17] showed that $n(5, 3, 4) = 13$ by detailed casework and this implies $n(6, 3, 5) \leqslant 14$; so the BCH code of Example 1 is optimal, and the extended code is optimal and gives $n(6, 3, 5) = 14$.

EXAMPLE 2 ($n = 13$; $n(8, 3, 6) \geqslant 13$; $n(9, 3, 7) \geqslant 14$). As in Example 1, let $g(x) = m_{\beta}(x)\, m_{\beta^2}(x)\, m_{\beta^4}(x) = x^9 + x^8 + 2x^7 + x^5 + 2x^3 + 2x^2 + 2$ and $h(x) = (x-1)\, m_{\beta^7}(x) = x^4 + 2x^3 + 2x^2 + 1$. Then, $g(x)$ generates a (13, 4)-linear code with designed distance 7 (which is the actual minimum distance). The orthogonal code is the code of a (13,6) set $C \subseteq PG(8, 3)$; that is, $n(8, 3, 6) \geqslant 13$. The extended code, in this case, gives a (14,7) set $\mathbf{C} \subseteq PG(9, 3)$; that is, $n(9, 3, 7) \geqslant 14$.

EXAMPLE 3 ($n = 20$; $4 \times 20 = 3^4 - 1$; $n(14, 3, 10) \geqslant 20$; $n(15, 3, 11) \geqslant 21$). Let β be a primitive 20th root of unity in $GF(3^4)$. Then, $x \to 3x$ acts on $\mathbb{Z}_{20}$ decomposing it into orbits

$$(0)(1, 3, 9, 7)(2, 6, 18, 14)(4, 12, 16, 8)(5, 15)(10)(11, 13, 19, 17),$$

and $g(x) = m_{\beta}(x) m_{\beta^2}(x)\, m_{\beta^4}(x)\, m_{\beta^5}(x)\, m_{\beta^{10}}(x) = x^{15} + 2x^{14} + x^{11} + x^9 + x^8 + x^7 + x^6 + 2x^4 + x^3 + x^2 + 1$ generates a (20, 5)-linear code with minimum distance 11. The orthogonal code is the code of a (20, 10) set

$C \subseteq PG(14, 3)$; that is, $n(14, 3, 10) \geqslant 20$. The extended code, in this case, gives a $(21, 11)$ set $\mathbf{C} \subseteq PG(15, 3)$; that is, $n(15, 3, 11) \geqslant 21$.

Next, two computer assisted searches are described and two applications of each given.

I. *Extending an (n, d) Set to a Maximal (n', d) Set*

Application 1 ($n(6, 3, 4) \geqslant 20$). Start with a $(14, 5)$ set $\subseteq PG(6, 3)$, and obtain a $(20, 4)$ set $\subseteq PG(6, 3)$ with matrix

$$H^{\tau} = \left[\begin{array}{cccccccccccccc} \underbrace{\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 0 \end{matrix}}_{\text{a (14,5)-set} \in PG(6,3)} \end{array} \begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 2 & 1 \end{matrix}\right].$$

The bounds $17 \leqslant n(6, 3, 4) \leqslant 33$ in [17] are thus improved to $20 \leqslant n(6, 3, 4) \leqslant 31$.

Application 2 ($n(7, 3, 4) \geqslant 32$). Start with the above $(20, 4)$ set $\subseteq PG(6, 3) \subseteq PG(7, 3)$, and obtain a $(32, 4)$ set $\subseteq PG(7, 3)$ with matrix

$$\left[\begin{array}{c|cccccccccccc} 0\ 0\ 0\ \cdots\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 2 \\ & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 \\ H^{\tau} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 \\ & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 1 \\ & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 0 & 2 & 1 \end{array}\right].$$

II. *Add to the Base Points*

To show that $n(r, q, d) \geqslant r + 1 + k$, find k points $A \subseteq PG(r, q)$ so that for $i = 1, 2, \ldots, \min(k, d)$, any linear combination of i of the points has weight $\geqslant d + 1 - i$. Let $H(A)$ denote the k by $r + 1$ matrix of the additional points, then

$$H = \begin{bmatrix} I_{r+1} \\ --- \\ H(A) \end{bmatrix}$$

is the matrix of a $(r+1+k, d)$ set $\subseteq PG(r, q)$. In practice, assume a certain pattern of zeros occurs in $H(A)$, and attempt to fill in the remaining entries of $H(A)$ with elements of $GF(q)\backslash\{0\}$. If successful, try to extend the result to $n(r+1, q, d+1) \geqslant r+2+k$ by adding a coordinate to each of the k additional points.

Application 1 (Gulati, [10])($n(7, 3, 6) \geqslant 11$; $n(8, 3, 7) \geqslant 12$, and $n(9, 3, 8) \geqslant 13$). Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 0 & 0 \end{bmatrix}; \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix};$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}; \qquad A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}.$$

Check that any linear combination of i rows, $i = 1, 2, 3$, of $A_{6+j} = [A \quad A_j]$, $j = 2, 3, 4$ has weight $\geqslant 5 + j - i$ so that

$$\begin{bmatrix} I_{6+j} \\ --- \\ A_{6+j} \end{bmatrix}$$

is the matrix of a $(9+j, 4+j)$ set $\subseteq PG(5+j, 3)$, $j = 2, 3, 4$. In Section 4, it was shown that

$$n(7, 3, 6) \leqslant 11; \quad n(8, 3, 7) \leqslant 12; \quad \text{and} \quad n(9, 3, 8) \leqslant 13,$$

which implies

$$n(7, 3, 6) = 11; \quad n(8, 3, 7) = 12; \quad \text{and} \quad n(9, 3, 8) = 13.$$

Application 2 ($n(12, 3, 9) \geqslant 17$; $n(13, 3, 10) \geqslant 18$; and $n(14, 3, 11) \geqslant 19$). Let

$$A_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$A_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}; \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}.$$

Check that any linear combination of i rows, $i = 1, 2, 3, 4$, of A_{13} or $A_{13+j} = [A_{13} \quad A_j]$, $j = 1, 2$, has weight $\geqslant 10 + j - i$, $j = 0, 1, 2$. Thus,

$$\begin{bmatrix} I_{13+j} \\ --- \\ A_{13+j} \end{bmatrix}$$

is the matrix of a $(17 + j,\ 9 + j)$ set $\subseteq PG(12 + j, 3)$, $j = 0, 1, 2$. In Section 4, it was shown that

$$n(12, 3, 9) \leqslant 18; \qquad n(13, 3, 10) \leqslant 19; \qquad \text{and} \qquad n(14, 3, 11) \leqslant 20,$$

which implies

$$17 \leqslant n(12, 3, 9) \leqslant 18, \qquad 18 \leqslant n(13, 3, 10) \leqslant 19,$$
$$19 \leqslant n(14, 3, 11) \leqslant 20.$$

Finally, Table VII summarizes the results for $n(r, 3, d)$ for $r \leqslant 15$. The portion of Table VII for $r \leqslant 5$ appears in Hill [17].

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